Функциональный интеграл в моделях гравитации

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Квадратичная гравитация

$$\mathcal{A} = \mathcal{A}_0 + \mathcal{A}_1 + \mathcal{A}_2 \tag{1}$$

$$\mathcal{A}_0 = \Lambda \int d^4x\sqrt{\mathcal{G}} \qquad \mathcal{A}_1 = -\frac{\kappa}{6}\int d^4x\sqrt{\mathcal{G}}R \tag{2}$$

$$\mathcal{A}_2 = \int d^4x\sqrt{\mathcal{G}}\left(c_1 R^2 + c_2 C_{\mu\nu\rho\lambda}C^{\mu\nu\rho\lambda}\right) \tag{3}$$

Метрика FLRW

$$ds^2 = N^2(t)dt^2 - a^2(t)d\mathbf{x}^2 \tag{4}$$

$$ds^2 = dt^2 - a^2(t)d\mathbf{x}^2 \tag{5}$$

$$ds^2 = (g'(\tau))^2(d\tau^2 - d\mathbf{x}^2) \tag{6}$$

$$t = g(\tau) \qquad a(t) = g'(g^{-1}(t)) \tag{7}$$

Одномерное действие

$$A = A_0 + A_1 + A_2 \qquad A_i = \frac{\mathcal{A}_i}{\int d^3x} \tag{8}$$

$$A_0 = \Lambda \int d\tau \, (g'(\tau))^4 \tag{9}$$

$$A_1 = -\kappa \int d\tau \left[(g''(\tau))^2 - \frac{d}{d\tau} (g'(\tau) g''(\tau)) \right] \tag{10}$$

$$A_2 = \frac{\lambda^2}{2} \int d\tau \left(\frac{g'''(\tau)}{g'(\tau)} \right)^2 \tag{11}$$

Решение классического уравнения движения при $\Lambda = 0$ есть

$$g(\tau) = \frac{\sigma\tau^2}{2} \qquad \tau(t) = \sqrt{\frac{2t}{\sigma}} \qquad a(t) = \sqrt{2\sigma t} \tag{12}$$

Функциональный интеграл

$$\int F(g)e^{-A(g)}dg = \int F(g)e^{-A_0(g)-A_1(g)}\mu(dg) \tag{13}$$

$$\mu(dg) = e^{-A_2(g)}dg \tag{14}$$

Введём переменную $q(\tau)$

$$q = \frac{g''}{g'} \qquad \frac{g'''}{g'} = q' + q^2 \tag{15}$$

Произведём нелинейную нелокальную замену переменных:

$$p(\tau) = q(\tau) + \int_0^\tau q^2(\tau')d\tau' \qquad \frac{g'''}{g'} = p' \tag{16}$$

При помощи этой замены мера (14) преобразуется к виду меры винера на пространстве кусочно-непрерывных функций $p(\tau)$

$$\mu(dg) = e^{-A_2(g)}dg = e^{-\int_0^\tau (p'(\tau_1))^2 d\tau_1}dg = w_{1/\lambda}(dp) \tag{17}$$

Винеровский интеграл

Интегралы по мере Винера от произведений функций $p(\tau)$ берутся по следующим правилам:

$$\int p(\tau) w_{1/\lambda}(dp) = 0 \qquad \int p(\tau_1) p(\tau_2) w_{1/\lambda}(dp) = \frac{1}{\lambda^2} \min(\tau_1, \tau_2) \quad (18)$$

В случае произведения большего количества сомножителей интеграл равен 0 при нечётном их числе, а при чётном может быть вычислен с помощью теоремы Вика.

$g(\tau)$ выразим через $p(\tau)$

$$\eta' = -(1 - p\eta)^2 \tag{19}$$

$$g(\tau) = -\sigma \int_0^\tau d\bar{\tau}\eta(\bar{\tau}) \exp\left(\int_0^{\bar{\tau}} d\tau_1 p(\tau_1)\left[1 - p(\tau_1)\eta(\tau_1)\right]\right) \tag{20}$$

Среднее значение масштабного фактора

$$< a(t) >= \frac{1}{Z} \int g'(\tau) e^{-A_1(\tau)} w_{1/\lambda}(dp) \tag{21}$$

$$A_1(\tau) = -\kappa \int_0^\tau d\tau_1 (g''(\tau_1))^2 + \kappa g'(\tau) g''(\tau) \tag{22}$$

$$\tau = g^{-1}(t) \text{ - тоже зависит от p} \tag{23}$$

теория возмущений:

$$< a(t) >= \sum_n \frac{\bar{a}_{2n}(t)}{\lambda^{2n}} \tag{24}$$

$$\frac{\bar{a}_2}{\lambda^2} = \int (E_1 a_1 + a_2) w_{1/\lambda}(dp) = \frac{\sigma}{\lambda^2} \left[-\frac{59}{63} \left(\frac{2t}{\sigma} \right)^2 + \frac{11}{120} \kappa \sigma^2 \left(\frac{2t}{\sigma} \right)^{5/2} \right] \tag{25}$$

$$a_2 \sim p^2 \qquad E_1,\ a_1 \sim p \tag{26}$$

Метод разложения "по степеням"p

$$p \to \alpha p \tag{27}$$

$$g(\tau, \alpha) = \sum_{k=0}^{\infty} g_k(\tau)\alpha^k \tag{28}$$

$$g(\tau(t, \alpha), \alpha) = t \tag{29}$$

$$a(t, \alpha) = g'(\tau(t, \alpha), \alpha) \tag{30}$$

$$A_1(t, \alpha) = A_1(\tau(t, \alpha), \alpha) \tag{31}$$

Пусть известны разложения функций

$$f(\tau, \alpha) = \sum_{k=0}^{\infty} f_k(\tau)\alpha^k \qquad \tau(t, \alpha) = \sum_{k=0}^{\infty} \tau_k(t)\alpha^k \tag{32}$$

И каждая из функций $f_k(\tau)$ разлагается в ряд Тейлора в окрестности точки $\tau_0(t)$

$$f_k(\tau) = \sum_{s=0}^{\infty} f_{ks}(\tau - \tau_0)^s \qquad f_{ks}(t) = \frac{1}{s!}\frac{d^s f}{d\tau^s}|_{\tau=\tau_0(t)} \tag{33}$$

Требуется вычислить коэффициенты $h_k(t)$ разложения:

$$h(t, \alpha) = f(\tau(t, \alpha), \alpha) = \sum_{k=0}^{\infty} h_k(t)\alpha^k \tag{34}$$

Ответ даётся формулой:

$$h_k(t) = \sum_{l=0}^{k} \sum_{s=0}^{k-l} f_{ls} T_{k-l}^{(s)} \tag{35}$$

$$T_m^{(s)} = \sum_{r=s-1}^{m-1} T_r^{(s-1)} \tau_{m-r} \qquad T_m^{(0)} = \delta_m^0 \tag{36}$$

$$a(t,\alpha) = g'(g^{-1}(t,\alpha),\alpha) = \sum_k a_k(t)\alpha^k \tag{37}$$

$$a_1 = g_{11} - \frac{2g_{02}}{g_{01}} g_{10} \tag{38}$$

$$a_2 = g_{21} - \frac{2g_{02}}{g_{01}} g_{20} - \frac{2g_{10}}{g_{01}} g_{12} + \frac{2g_{02}}{g_{01}^2} g_{10} g_{11} + \frac{3g_{03}}{g_{01}^2} g_{10}^2 - \frac{2g_{02}^2}{g_{01}^3} g_{10}^2 \tag{39}$$

$$
\begin{aligned}
a_3 = g_{31} - \frac{2g_{02}}{g_{01}} g_{30} - \frac{2g_{10}}{g_{01}} g_{22} - \frac{2g_{12}}{g_{01}} g_{20} + \frac{2g_{02}}{g_{01}^2} g_{10} g_{21} + \frac{2g_{02}}{g_{01}^2} g_{11} g_{20} + \\
+ \frac{6g_{03}}{g_{01}^2} g_{10} g_{20} + \frac{3g_{13}}{g_{01}^2} g_{10}^2 + \frac{2g_{10}}{g_{01}^2} g_{11} g_{12} - \frac{4g_{10}}{g_{01}^3} g_{02}^2 g_{20} - \frac{4g_{02}}{g_{01}^3} g_{10}^2 g_{12} - \\
- \frac{2g_{02}}{g_{01}^3} g_{10} g_{11}^2 - \frac{6g_{03}}{g_{01}^3} g_{10}^2 g_{11} - \frac{4g_{04}}{g_{01}^3} g_{10}^3 + \frac{6g_{11}}{g_{01}^4} g_{02}^2 g_{10}^2 + \frac{8g_{02}}{g_{01}^4} g_{03} g_{10}^3 - \frac{4g_{02}^3}{g_{01}^5} g_{10}^3
\end{aligned}
\tag{40}
$$

$$
\begin{aligned}
a_4 = {} & g_{41} - \frac{2g_{02}}{g_{01}} g_{40} - \frac{2g_{10}}{g_{01}} g_{32} - \frac{2g_{12}}{g_{01}} g_{30} - \frac{2g_{20}}{g_{01}} g_{22} + \frac{2g_{02}}{g_{01}^2} g_{10} g_{31} + \\
& + \frac{2g_{02}}{g_{01}^2} g_{11} g_{30} + \frac{2g_{02}}{g_{01}^2} g_{20} g_{21} + \frac{6g_{03}}{g_{01}^2} g_{10} g_{30} + \frac{3g_{03}}{g_{01}^2} g_{20}^2 + \frac{3g_{23}}{g_{01}^2} g_{10}^2 + \\
& + \frac{2g_{10}}{g_{01}^2} g_{11} g_{22} + \frac{2g_{10}}{g_{01}^2} g_{12} g_{21} + \frac{6g_{10}}{g_{01}^2} g_{13} g_{20} + \frac{2g_{11}}{g_{01}^2} g_{12} g_{20} - \frac{4g_{10}}{g_{01}^3} g_{02}^2 g_{30} - \frac{2g_{02}^2}{g_{01}^3} g_{20}^2 - \\
& - \frac{4g_{02}}{g_{01}^3} g_{10}^2 g_{22} - \frac{4g_{02}}{g_{01}^3} g_{10} g_{11} g_{21} - \frac{8g_{02}}{g_{01}^3} g_{10} g_{12} g_{20} - \frac{2g_{02}}{g_{01}^3} g_{11}^2 g_{20} - \\
& - \frac{6g_{03}}{g_{01}^3} g_{10}^2 g_{21} - \frac{12g_{03}}{g_{01}^3} g_{10} g_{11} g_{20} - \frac{12g_{04}}{g_{01}^3} g_{10}^2 g_{20} - \frac{4g_{14}}{g_{01}^3} g_{10}^3 - \frac{6g_{11}}{g_{01}^3} g_{10}^2 g_{13} - \frac{2g_{10}^2}{g_{01}^3} g_{12}^2 - \\
& - \frac{2g_{10}}{g_{01}^3} g_{11}^2 g_{12} + \frac{6g_{21}}{g_{01}^4} g_{02}^2 g_{10}^2 + \frac{12g_{10}}{g_{01}^4} g_{02}^2 g_{11} g_{20} + \frac{24g_{02}}{g_{01}^4} g_{03} g_{10}^2 g_{20} + \\
& + \frac{8g_{02}}{g_{01}^4} g_{10}^3 g_{13} + \frac{12g_{02}}{g_{01}^4} g_{10}^2 g_{11} g_{12} + \frac{2g_{02}}{g_{01}^4} g_{10} g_{11}^3 + \frac{8g_{03}}{g_{01}^4} g_{10}^3 g_{12} + \frac{9g_{03}}{g_{01}^4} g_{10}^2 g_{11}^2 + \\
& + \frac{12g_{04}}{g_{01}^4} g_{10}^3 g_{11} + \frac{5g_{05}}{g_{01}^4} g_{10}^4 - \frac{12g_{20}}{g_{01}^5} g_{02}^3 g_{10}^2 - \frac{12g_{12}}{g_{01}^5} g_{02}^2 g_{10}^3 - \frac{12g_{02}^2}{g_{01}^5} g_{10}^2 g_{11}^2 - \\
& - \frac{32g_{02}}{g_{01}^5} g_{03} g_{10}^3 g_{11} - \frac{14g_{02}}{g_{01}^5} g_{04} g_{10}^4 - \frac{6g_{03}^2}{g_{01}^5} g_{10}^4 + \frac{20g_{11}}{g_{01}^6} g_{02}^3 g_{10}^3 + \frac{25g_{03}}{g_{01}^6} g_{02}^2 g_{10}^4 - \frac{10g_{02}^4}{g_{01}^7} g_{10}^4
\end{aligned}
$$

$$I_w(f) = \left(\int f[p] w_{1/\lambda}(dp) \right) |_{\lambda=1} \tag{42}$$

$$\bar{a}_2 = I_w(a_1 E_1) + I_w(a_2) \tag{43}$$

$$\bar{a}_4 = I_w(E_2)\left(a_0 I_w(E_2) - I_w(a_1 E_1)\right) + \left(I_w(a_2 E_2) - I_w(a_2) I_w(E_2)\right) + \\ + I_w(a_3 E_1) + I_w(a_1 E_3) + I_w(a_4) \tag{44}$$

Пример

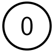
$$f[p] = \int_0^\beta d\tau_1 p(\tau_1) \int_0^{\tau_1} d\tau_2 \tau_2 p(\tau_2) \qquad \beta(t) \equiv \tau_0(t) = \sqrt{\frac{2t}{\sigma}} \tag{45}$$

$$J(\beta) = I_w(f) = \int_0^\beta d\tau_1 \int_0^{\tau_1} d\tau_2 \tau_2 \min(\tau_1, \tau_2) \tag{46}$$

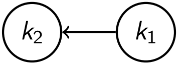
$$J'(\beta) = \int_0^\beta d\tau_2 \tau_2 \min(\beta, \tau_2) = \int_0^\beta d\tau_2 (\tau_2)^2 = \frac{\beta^3}{3} \tag{47}$$

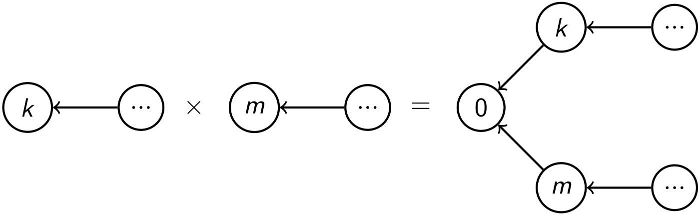
$$J(\beta) = \frac{\beta^4}{12} \tag{48}$$

Построение структуры данных

$$p(\tau) \backsimeq$$  (49)

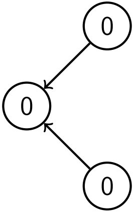
$$\tau^k p(\tau) \backsimeq$$  (50)

$$\tau^{k_2} \int_0^{\tau} d\tau_1 \tau_1^{k_1} p(\tau_1) \backsimeq$$  (51)



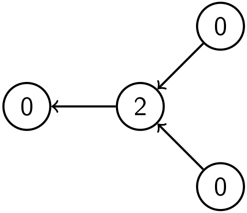
(52)

$$p^2(\tau) \backsimeq$$



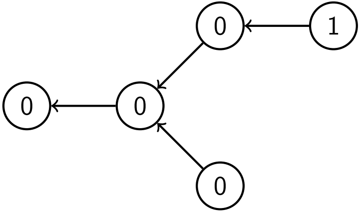
(53)

$$\int_0^\tau d\tau_1 \tau_1^2 p^2(\tau_1) \backsimeq$$



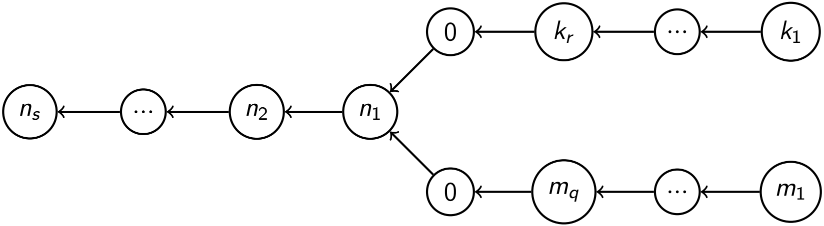
(54)

$$\int_0^\tau d\tau_1 p(\tau_1) \left(\int_0^{\tau_1} d\tau_2 \tau_2 p(\tau_2) \right) \backsimeq$$



(55)

Рекуррентная формула для $I_w(f)$



(56)

$$I_w(f) = b(k_r, \dots, k_1 | m_q, \dots, m_1) \left(\prod_{j=1}^{s-1} \frac{1}{S' + j + \sum_{l=1}^{j} n_l} \right) \beta^{\sum_{l=1}^{s} n_l + s - 1 + S'} \tag{57}$$

$$S' = \sum_{l=1}^{r} k_l + \sum_{l=1}^{q} m_l + r + q + 1 \tag{58}$$

$$b(k_r, ..., k_1 | m_q, ..., m_1) = \frac{b(k_{r-1}, ..., k_1 | m_q, ..., m_1) + b(k_r, ..., k_1 | m_{q-1}, ..., m_1)}{\sum_{l=1}^{r} k_l + \sum_{l=1}^{q} m_l + r + q + 1} \quad (59)$$

$$b(\,|\,) = 1 \quad (60)$$

$$
\begin{aligned}
\frac{a_1 E_1}{\kappa\sigma^3} = {} & 2\beta \int_0^\beta d\tau_1 p(\tau_1) \int_0^\beta d\tau_1 \tau_1 p(\tau_1) - 3\beta \int_0^\beta d\tau_1 p(\tau_1) \beta^2 p(\beta) + \\
& + 4 \int_0^\beta d\tau_1 \tau_1 p(\tau_1) \int_0^\beta d\tau_1 \tau_1 p(\tau_1) - 6 \int_0^\beta d\tau_1 \tau_1 p(\tau_1) \beta^2 p(\beta) - \\
& - 2\beta^{-1} \int_0^\beta d\tau_1 \tau_1 \int_0^{\tau_1} d\tau_2 p(\tau_2) \int_0^\beta d\tau_1 \tau_1 p(\tau_1) + \\
& + 3\beta^{-1} \int_0^\beta d\tau_1 \tau_1 \int_0^{\tau_1} d\tau_2 p(\tau_2) \beta^2 p(\beta) - \\
& - 4\beta^{-1} \int_0^\beta d\tau_1 \int_0^{\tau_1} d\tau_2 \tau_2 p(\tau_2) \int_0^\beta d\tau_1 \tau_1 p(\tau_1) + \\
& + 6\beta^{-1} \int_0^\beta d\tau_1 \int_0^{\tau_1} d\tau_2 \tau_2 p(\tau_2) \beta^2 p(\beta)
\end{aligned} \tag{61}
$$

$$
I_w(a_1 E_1) = -\frac{139}{72} \beta^5 \kappa\sigma^3 \tag{62}
$$

f	$I_w(f)$
$\beta \int_0^\beta d\tau_1 p(\tau_1) \int_0^\beta d\tau_1 \tau_1 p(\tau_1)$	$\frac{5}{24}\beta^5$
$\beta \int_0^\beta d\tau_1 p(\tau_1)\beta^2 p(\beta)$	$\frac{1}{2}\beta^5$
$\int_0^\beta d\tau_1 \tau_1 p(\tau_1) \int_0^\beta d\tau_1 \tau_1 p(\tau_1)$	$\frac{2}{15}\beta^5$
$\int_0^\beta d\tau_1 \tau_1 p(\tau_1)\beta^2 p(\beta)$	$\frac{1}{3}\beta^5$
$\beta^{-1} \int_0^\beta d\tau_1 \tau_1 \int_0^{\tau_1} d\tau_2 p(\tau_2) \int_0^\beta d\tau_1 \tau_1 p(\tau_1)$	$\frac{1}{18}\beta^5$
$\beta^{-1} \int_0^\beta d\tau_1 \tau_1 \int_0^{\tau_1} d\tau_2 p(\tau_2)\beta^2 p(\beta)$	$\frac{1}{8}\beta^5$
$\beta^{-1} \int_0^\beta d\tau_1 \int_0^{\tau_1} d\tau_2 \tau_2 p(\tau_2) \int_0^\beta d\tau_1 \tau_1 p(\tau_1)$	$\frac{13}{360}\beta^5$
$\beta^{-1} \int_0^\beta d\tau_1 \int_0^{\tau_1} d\tau_2 \tau_2 p(\tau_2)\beta^2 p(\beta)$	$\frac{1}{12}\beta^5$

$$I_w(a_2) = \frac{18}{35}\beta^4\sigma \tag{63}$$

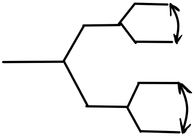
$$I_w(a_1 E_1) = -\frac{139}{72}\beta^5\kappa\sigma^3 \tag{64}$$

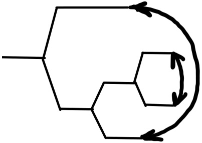
$$I_w(E_2) = -\frac{9}{2}\beta^4\kappa\sigma^2 + \frac{83}{30}\beta^5(\kappa\sigma^2)^2 \tag{65}$$

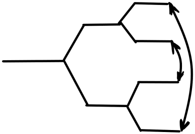
$$\bar{a}_2 = -\frac{139}{72}\beta^5\kappa\sigma^3 + \frac{18}{35}\beta^4\sigma \tag{66}$$

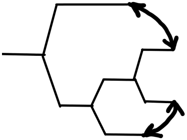
$$a_4 = \frac{6889}{900}\beta^{11}\kappa^4\sigma^9 + c_7\beta^7\sigma + c_8\beta^8\kappa\sigma^3 + c_9\beta^9\kappa^2\sigma^5 + c_{10}\beta^{10}\kappa^3\sigma^7 \tag{67}$$

Спасибо за внимание!









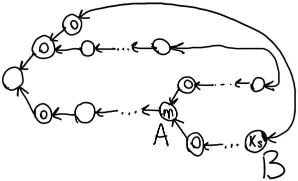


Рис.: