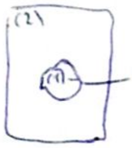
Лекция 4

Свободная энергия для канонического распределения



V, T, N - const. $\quad E \neq$ const

$$F = E - TS$$

$$\delta R = dF$$

изменение св. энергии = = работа при бескон. малом изотерм. изменении состоян.

$$dF = -S\,dT - P\,dV$$

$$S = -\left(\frac{\partial F}{\partial T}\right)_V, \quad P = -\left(\frac{\partial F}{\partial V}\right)_T$$

$$T, V - \text{const} \Rightarrow F = \text{const} = F(V, T)$$

$\cancel{F \neq \frac{F}{V}} = f(T)$ - точно определено для т/д состояния.

$$Z = \mathrm{Tr}\left(e^{-\beta \hat{H}}\right) = e^{-\beta F} = \int_{\text{p.b.c.}} Dx\, e^{-\overset{(\beta)}{S}[x]}$$

$S[x]$ — $\frac{m\dot{x}^2}{2} + V(x)$

$$E = \langle H \rangle = \frac{1}{Z}\mathrm{Tr}\left(\hat{H} e^{-\beta \hat{H}}\right)$$

$$S = \langle \ln \hat{\rho} \rangle$$

$$\hat{\rho} = \frac{1}{Z} e^{-\beta \hat{H}}$$

$$\ln \hat{\rho} = -\beta \hat{H} - \ln Z$$

$$S = \mathrm{Tr}(-\ln \hat{\rho}) = \beta \langle H \rangle + \ln Z$$

$$S = \beta E - \beta F \qquad F = -T \ln Z$$

Терм. теория поля $\Rightarrow$ периодичность по фиктивному евклид. "времени", [1] (причем $\beta \equiv 1/T$)

бозоны | поля коммутир. переменные $\Rightarrow$ периодич. гран. условия $\phi(\tau + \beta) = \phi(\tau)$ $\rightarrow$

фермионы | -"- антикоммутир. $\Rightarrow$ антипериодич. гран. условия $\psi(\tau + \beta) = -\psi(\tau)$ $\rightarrow$

$$\Rightarrow \omega_n = \frac{2\pi n}{\beta}, \quad n \in \mathbb{Z}$$

$$\rightarrow \omega_{n'} = \frac{2\pi n'}{\beta}, \quad n = \pm\frac{1}{2}, \pm\frac{3}{2} \ldots$$

вместо конт. Фурье-обр. по времени ряд по дискретным частотам ω_n

$$Z = e^{-\beta F} = \int D\varphi\, e^{-S^{(\beta)}[\varphi]} = \left[\det\left(-\Box_\beta + m^2\right)\right]^{-1/2}$$

$$F = T \cdot \frac{1}{2} \ln \det \underbrace{\left(-\Box_\beta + m^2\right)}_{\hat{M}} \qquad \omega_n = \frac{2\pi}{\beta} n$$

$$\det \hat{M} = \prod_n \lambda_n \qquad \lambda_{n,\vec{p}} = \omega_n^2 + \vec{p}^{\,2} + m^2$$

ящик L^3 $\quad \vec{p} = \left(\frac{2\pi n_1}{L}, \frac{2\pi n_2}{L}, \frac{2\pi n_3}{L}\right)$

$$\det \hat{M} = \prod_n \prod_{n_1} \prod_{n_2} \prod_{n_3} \lambda_{n,n_1,n_2,n_3} \qquad \lambda_{n,n_1,n_2,n_3} = \omega_n^2 + \vec{p}^{\,2}_{n_1,n_2,n_3} + m^2$$

$$F = \frac{T}{2} \sum_n \sum_{n_1,n_2,n_3} \ln\left(\frac{\vec{p}^{\,2} + \omega_n^2 + m^2}{\Lambda^2}\right) \longmapsto L^3 \cdot \frac{T}{2} \sum_n \int \frac{d^3p}{(2\pi)^3} \ln\left(\frac{\vec{p}^{\,2} + \omega_n^2 + m^2}{\Lambda^2}\right)$$

плотность свободной энергии $\quad f = \frac{F}{L^3}$

$$f = \frac{T}{2} \sum_n \int \frac{d^3p}{(2\pi)^3} \ln\left(\frac{\vec{p}^{\,2} + \omega_n^2 + m^2}{\Lambda^2}\right)$$

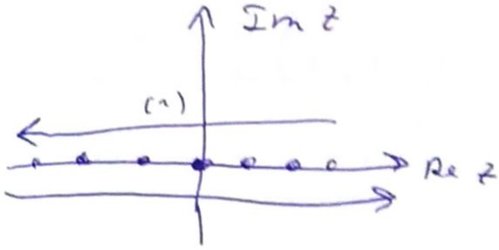
удобнее посчитать $\frac{\partial f}{\partial m^2}$, потом проинтегр. по m^2

$$\frac{\partial f}{\partial m^2} = \frac{T}{2} \int \frac{d^3p}{(2\pi)^3} \sum_n \frac{1}{\vec{p}^{\,2} + \omega_n^2 + m^2}$$

используем трюк:

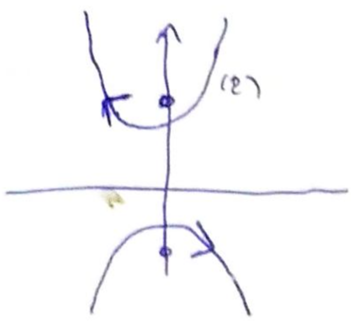
$$\sum_{n \in \mathbb{Z}} u(n) = \frac{1}{2i} \oint \operatorname{ctg}(\pi z)\, u(z)\, dz$$

полюса – целые числа z
вычет – π^{-1}



$$u(z) = \left[\vec{p}^{\,2} + \left(\frac{2\pi}{\beta} z\right)^2 + m^2\right]^{-1}$$

если $u(z)$ спадает $|z| \to \infty$ лучше чем $\frac{1}{z}$, деформируем контур.



Полюса $u(z)$:

$$z = \pm \frac{i\beta}{2\pi} \sqrt{\vec{p}^{\,2} + m^2}$$

$$\frac{\partial f}{\partial m^2} = \frac{1}{2} \int \frac{d^3p}{(2\pi)^3} \oint \frac{dz}{2i} \operatorname{ctg}(\pi z) \frac{1}{\vec{p}^{\,2} + \left(\frac{2\pi}{\beta} z\right)^2 + m^2}$$

берем вычеты по контуру (2)

$$\frac{\partial f}{\partial m^2} = \frac{1}{2\cancel{\beta}}\int\frac{d^3p}{(2\pi)^3}\oint\frac{dz}{2i}\,\mathrm{ctg}\,\pi z\,\frac{1}{\left(\frac{2\pi}{\beta}\right)^2\left(z+\frac{i\beta}{2\pi}\sqrt{\vec{p}^2+m^2}\right)\left(z-\frac{i\beta}{2\pi}\sqrt{\vec{p}^2+m^2}\right)}$$

вычет в $z=\frac{i\beta}{2\pi}\sqrt{\vec{p}^2+m^2}$

$$\frac{\partial f}{\partial m^2} = \frac{1}{2\cancel{\beta}}\cdot\left(\frac{\beta}{2\pi}\right)^2\cdot 2\int\frac{d^3p}{(2\pi)^3}\,\frac{2\pi i}{2i}\,\frac{\mathrm{ctg}\left(i\frac{\beta}{2}\sqrt{\vec{p}^2+m^2}\right)}{\frac{i\beta}{\pi}\sqrt{\vec{p}^2+m^2}} =$$

$$= \cancel{\beta}\int\frac{d^3p}{(2\pi)^3}\,\frac{\mathrm{cth}\left(\frac{\beta}{2}\sqrt{\vec{p}^2+m^2}\right)}{4\sqrt{\vec{p}^2+m^2}}$$

при $T\to 0$ $\beta\to\infty$, $\mathrm{cth}\to 1$

расх. вклад при $T=0$, выделим.

$$\frac{\partial f}{\partial m^2} = \underbrace{\beta\int\frac{d^3p}{(2\pi)^3}\,\frac{1}{4\sqrt{\vec{p}^2+m^2}}}_{T=0} + \beta\int\frac{d^3p}{(2\pi)^3}\,\frac{\mathrm{cth}\left(\frac{\beta}{2}\sqrt{\vec{p}^2+m^2}\right)-1}{4\sqrt{\vec{p}^2+m^2}}$$

энергия нулевых колебаний. перенормировка.

$$\Big/\!\Big/\ \mathrm{cth}\,a - 1 = \frac{\mathrm{ch}\,a-\mathrm{sh}\,a}{\mathrm{sh}\,a} = \frac{e^{a}+e^{-a}-e^{a}+e^{-a}}{e^{a}-e^{-a}} = \frac{2e^{-a}}{e^{a}-e^{-a}} = \frac{2}{e^{2a}-1}\ \Big/\!\Big/$$

$$\frac{\partial f^{(T)}}{\partial m^2} = \cancel{\beta}\int\frac{d^3p}{(2\pi)^3}\,\frac{1}{2\sqrt{\vec{p}^2+m^2}}\,\frac{1}{e^{\sqrt{\vec{p}^2+m^2}/T}-1} = \cancel{\beta}\int_0^\infty\frac{p^2dp}{4\pi^2\sqrt{p^2+m^2}}\,\frac{1}{e^{\sqrt{p^2+m^2}/T}-1}$$

$$\cancel{f^{(T)} = -\frac{1}{6\pi^2}\int_0^\infty\frac{p^4dp}{\sqrt{p^2+m^2}}\,\frac{1}{e^{\sqrt{p^2+m^2}/T}-1}}$$

$$\frac{1}{e^{x}-1} = \frac{d}{dx}\ln\left(1-e^{-x}\right)$$

$$\frac{d}{dm^2}\ln\left(1-e^{-\sqrt{\vec{p}^2+m^2}/T}\right) = \frac{1}{e^{\sqrt{\vec{p}^2+m^2}/T}-1}\cdot\frac{1}{2\sqrt{\vec{p}^2+m^2}}\cdot\frac{1}{T}$$

$$f^{(T)} = T\int\frac{d^3p}{(2\pi)^3}\ln\left(1-e^{-\sqrt{\vec{p}^2+m^2}/T}\right)$$

$f^{(T)} = -P$ — давление.

вычислим при $p\gg m$

$$P = -T\int\frac{p^2dp}{2\pi^2}\ln\left(1-e^{-p/T}\right) = \frac{-T^4}{2\pi^2}\underbrace{\int_0^\infty x^2dx\,\ln\left(1-e^{-x}\right)}_{-\pi^4/45} = \frac{\pi^2T^4}{90}$$

Свободная энергия фермионного поля.

$$e^{-F} = \int D\bar\psi\, D\psi\, e^{-S_\psi^{(\beta)}} = \det\left[\gamma^\mu \partial_\mu + m\right]$$

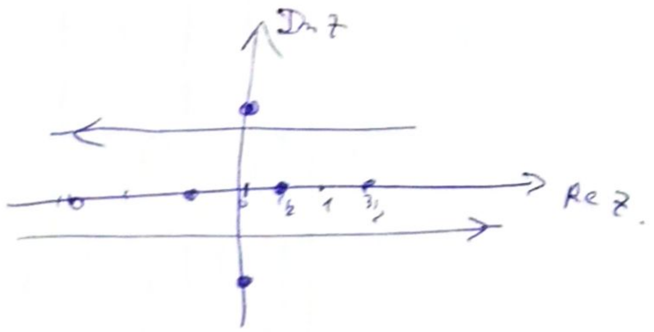
|| $\gamma \to -\gamma$ ничего не изменит

$$\lambda_\pm = m \pm i\sqrt{p^2 + \cancel{\omega_n^2}}$$

$$\omega_{n'} = \frac{2\pi n'}{\beta}$$

$$f_\psi = -\frac{2}{\beta}\int \frac{d^3p}{(2\pi)^3} \sum_{n' = \pm\frac{1}{2}, \pm\frac{3}{2}, \ldots} \ln\left(\frac{p^2 + \omega_n^2 + m^2}{n^2}\right)$$

$$\sum_{n' = \pm\frac{1}{2}, \pm\frac{3}{2}, \ldots} u(n') = \frac{1}{2i}\oint tg(\pi z)\, u(z)\, dz$$



$$\frac{\partial f_\psi}{\partial m^2} = -\frac{2}{\beta}\int \frac{d^3p}{(2\pi)^3} \cdot \frac{1}{2i}\oint dz\, tg(\pi z)\, \frac{1}{\omega_{n'}^2 + \vec p^2 + m^2} =$$

$$= -\frac{2}{\beta} \cdot 2 \cdot \left(\frac{\beta}{2\pi}\right)^2 \int \frac{d^3p}{(2\pi)^3} \cdot \frac{tg\left(\pi \cdot i\frac{\beta}{2\pi}\sqrt{\vec p^2 + m^2}\right)}{i \cdot \frac{\beta}{\pi}\sqrt{\vec p^2 + m^2}} \cdot \frac{2\pi i}{i} =$$

$$= -2\int \frac{d^3p}{(2\pi)^3} \cdot \frac{th\left(\frac{\beta}{2}\sqrt{\vec p^2 + m^2}\right)}{\sqrt{\vec p^2 + m^2}} =$$

// ~~[illegible]~~ $\beta = \infty \Rightarrow T = 0$, $th = 1$

$$= \underbrace{-2\int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{\vec p^2 + m^2}}}_{A} + 2\int \frac{d^3p}{(2\pi)^3} \frac{1 - th\left(\frac{\beta}{2}\sqrt{\vec p^2 + m^2}\right)}{\sqrt{\vec p^2 + m^2}}$$

// $1 - th\, a = \frac{ch\, a - sh\, a}{ch\, a} = \frac{e^a + e^{-a} - e^a + e^{-a}}{e^a + e^{-a}} = \frac{2e^{-a}}{e^a + e^{-a}} = \frac{2}{e^{2a} + 1}$

$$\frac{\partial f_\psi^{(T)}}{\partial m^2} = 4\int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{\vec p^2 + m^2}} \frac{1}{e^{\sqrt{\vec p^2 + m^2}/T} + 1}$$

$$\cancel{f^{(T)} = \frac{1}{6\pi^2}\int_0^\infty \frac{p^4\, dp}{\sqrt{p^2 + m^2}} \frac{1}{e^{\sqrt{p^2 + m^2}/T} + 1}}$$

← стат. добавки

$$\frac{1}{e^x + 1} = -\frac{d}{dx}\ln\left(1 + e^{-x}\right)$$

$$f^{(T)} = -\int \frac{d^3p}{(2\pi)^3}\, T \ln\left(1 + e^{-\sqrt{\vec p^2 + m^2}/T}\right)$$

Хим. потенциал.

Большой канонический ансамбль.

число частиц (заряд) $\neq$ const

$$\hat{H} \mapsto \hat{H} - \mu \hat{Q}$$

$$Q = \int d^3x \, j_0 = \int d^3x \, \bar{\psi} \gamma^0 \psi$$

$$j_\mu = i \bar{\psi} \gamma^\mu \psi$$

$$S_E^{(\beta)} \to S_E^{(\beta)} - \mu \int_0^\beta d\tau \int d^3x \, \bar{\psi} \gamma^0 \psi$$

$$\mathcal{L}_E = \bar{\psi} \gamma_\mu \partial_\mu \psi + m \bar{\psi} \psi - \mu \bar{\psi} \gamma^0 \psi = \bar{\psi} \left((\gamma_0 \partial_0 - \mu \gamma_0) + \gamma_i \partial_i \right) \psi + m \bar{\psi} \psi$$

$$e^{-\beta \Omega} = Z = \int_{\text{a.p.b.c.}} D\bar{\psi} D\psi \, \exp\left[-S_E^{(\beta,\mu)} \right]$$

$\gamma_0 \ldots \omega_n$

$\partial_0 \to \partial_0 - \mu$

$ip_0 \to ip_0 - \mu$

$p_0 \to p_0 + i\mu$

$\omega_n \to \omega_n + i\mu$

$$\hat{\rho} = \frac{e^{-\beta(\hat{H} - \mu \hat{Q})}}{Z}, \quad \operatorname{Tr} \hat{\rho} = 1$$

$$Z = \operatorname{Tr} e^{-\beta(\hat{H} - \mu \hat{Q})}$$

$$Z = e^{-\beta(F - \mu Q)}$$

Ω

$$\frac{1}{(\omega_n + i\mu)^2 + p^2 + m^2}$$

$\ldots$

$$\Omega = F - \mu Q = E - TS - \mu Q$$

$$\langle Q \rangle = -\frac{\partial \Omega}{\partial \mu}$$

HW.

$$\frac{\Omega^{(T)}}{V} = -2T \int \frac{d^3k}{(2\pi)^3} \left[\ln\left(1 + e^{-\frac{\sqrt{p^2+m^2} - \mu}{T}}\right) + \ln\left(1 + e^{-\frac{\sqrt{p^2+m^2} + \mu}{T}}\right) \right]$$